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FOURTH QUARTERLY PROGRESS REPORT

FOR

"CERTAIN COMPUTER PROGRAMS"

Contract No. NAS5-9700

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SECTION 1

ADVANCED ERROR PROPAGATION PROGRAM

The fourth quarter's progress on the Advanced Error Propagation Program (AEPP) consisted mainly of programming of the major part of the program, and led into the final checkout phase. That is, the program is now capable of generating numerical data, but these data must be examined in detail to verify programming and theoretical correctness. The numerous options of the program still require checkout, and in some cases, completion in detail.

Programming effort has been concentrated in three areas. These areas are 1) precision trajectory and targetting, 2) tape read and write, and 3) patched-conic error propagation. The first of these efforts is described in Section 1.1 and includes, in addition to integration of a perturbed lunar or interplanetary trajectory, the refinement of launch conditions so that the precision trajectory will satisfy prescribed target constraints. Section 1.2 describes the method used for writing the trajectory on tape so that the tape may be used to provide the reference trajectory for error propagation. Further effort has been put toward writing error propagation output information on tape for plotting and other purposes. Section 1.3 describes the state of completion and checkout of the Patched Conic Error Propagation Program (PCEPP). This program, which is a complete program in itself, was developed as a framework for checkout of the input-output, error propagation and measurement processing subroutines for the AEPP.

Appendix A contains the derivations of the "measurement partials" or sensitivities of the various measurements to deviations in the expanded state (vehicle position and velocity, station location and time bias errors). Significant savings have been made in the number of machine operations per measurement partial, relative to the original error propagation program delivered to Goddard Space Flight Center. In addition, some measurement rate partials which had not previously been included have been derived.

Measurement sensitivities to station location errors are now calculated relative to east, north and up errors (Cartesian) rather than assuming latitude and longitude and altitude errors and having to convert standard deviations from meters to radians.

1.1 TARGETTING CAPABILITY

A capability has been developed, under Contract NAS5-9700, for correcting injection conditions so that the resulting precision trajectory will satisfy a specified set of target constraints. The program which provides this capability has been checked out for Earth-to-Moon and Earth-to-Mars cases and is currently being checked out for the Earth return case.

Assumptions for this program are essentially those of the Quick Look Mission Analysis Program (reference 1) regarding launch controls and target constraints. The user provides the latitude, longitude, altitude, velocity azimuth and date of park orbit insertion. Starting values for park time and energy of the transfer trajectory are presumed to have been obtained from the Quick Look Program. Target conditions are specified in terms of radius of closest approach to the target, longitude and latitude of a point to be contained in the approach plane and, by option, flight time or target-relative energy.

1.1.1 Discussion of the Targetting Method

At initiation of the targetting process, and for each successive iteration, sensitivity of the injection state (position and velocity) to the various controls (launch time, park duration, injection energy, for example) is found by differences. Mathematically, if we denote the injection state by X(0) and the control vector by Q, we have

$$X(0) = F(Q).$$

* Reference 1 Programmer's Manual for the Quick Look Program--written for Goddard Space Flight Center under Contract NAS5-3342.

That is, the injection state is a function of the controls, this function being embodied in subroutine BEGIN. Each column of the sensitivity matrix, $\frac{\partial X(0)}{\partial Q}$, is computed by the approximation

$$\frac{\partial X(0)}{\partial Q_i} \approx \frac{F(Q + \delta Q_i) - F(Q)}{\delta Q_i}$$

where δQ_i is a small deviation in the ith control.

Linear propagation of these sensitivities to the target is performed by integration of the set of variational equations

$$\frac{d}{dt}\left(\frac{\partial X}{\partial Q}\right) = \left(\frac{\partial \dot{X}}{\partial X}\right)\frac{\partial X}{\partial Q}$$

subject to the initial conditions, $\frac{\partial X(0)}{\partial Q}$. The forcing matrix, $\frac{\partial \dot{X}}{\partial X}$, is analytically evaluated along the current trajectory for each iteration. The sensitivity of end conditions (constraints) to controls is also found by differences. Target constraint functions, ϕ, depend upon the final state, $X(t_f)$, in the following way,

$$\phi = G\left[X(t_f)\right]$$

where the functional relationship is provided by subroutine BODCON which accepts X as an input and outputs ϕ. Each column of the constraints/control sensitivity matrix is computed by

$$\frac{\partial \phi}{\partial Q_i} \approx \frac{G\left[X(t_f) + K\,\frac{\partial X(t_f)}{\partial Q_i}\right] - G\left[X(t_f)\right]}{K}$$

where K is a scale factor which limits the state end point position deviation in order to eliminate non-linearity problems.

New control values are computed from the constraint/control sensitivity matrix and the error in the constraint vector. Computation of these new values is performed by subroutine FNDMXN and amounts to the relationship

$$\delta Q = -\left(\frac{\partial \phi}{\partial Q}\right)^{-1} \delta\phi$$

when $\frac{\partial \phi}{\partial Q}$ is invertible. FNDMXN may also be used to find extrema of functions subject to other constraints (see Reference 1).

1.1.2 Trajectory Options

If the user wishes to use the targetting option, he may choose his control variables from the set:

1) time of launch
2) park duration
3) azimuth of the injection impulsive velocity vector
4) path angle of the injection impulsive velocity vector
5) energy at injection.

The available controls for any particular case are (1,2) or (1,2,5) for "earth-fixed" park orbits and (2,3,4), (2,3,4,5) or (3,4,5) for "inertial" park orbits. The user may choose to constrain flight time, target-relative energy, or neither, and in the inertial park orbit case, may minimize the magnitude of the injection velocity impulse. Controls and injection conditions may be stored so that successive cases can be run for refining to different end conditions, obtaining a detailed output time history, or generating a binary tape of the trajectory (see section 1.2).

If the targetting option is not selected, the user may start the integration by inputting the injection conditions in terms of orbital elements, spherical coordinates or Cartesian components in the equator and equinox of 1950 (or of date) system.

The program will then integrate to a stop time or specified radius from the target, writing detailed output and/or an ephemeris tape.

The perturbing influences which are currently in effect in calculation of the precision trajectory are attractions by all non-central bodies in the ephemeris and earth oblateness. Lunar oblateness, longitudinal harmonics of the earth's gravitational field, solar pressure and an approximate atmospheric drag effect will be included before the program is delivered. An Encke method is used for integrating the perturbing accelerations about a reference conic section, with rectification when the trajectory comes within the sphere of influence of another attracting body. An integration package, DEQ, developed under this contract, is being used to perform the numerical integration.

1.2 CAPABILITY FOR GENERATING AN EPHEMERIS TAPE

A group of subroutines has been written and checked out for writing interpolation coefficients of the simulated vehicle ephemeris and other relevant trajectory information on a binary tape and for reading and interpolating to recover the information from the tape. This package has been used with the precision trajectory program discussed in section 1.1 and has proven successful for earth-moon trajectories.

Use of a vehicle ephemeris tape with the AEPP will provide the necessary values of position and velocity at any desired frequency with advantages in machine run time and storage requirements. The principal need for position and velocity in error propagation arises in computing measurement sensitivities to the state. This need arises each time a measurement to be processed is taken, and this might occur much more frequently than one would choose to stop in an integrated or patched conic trajectory calculation. The AEPP will also require position and velocity to form the acceleration function for integrating variational equations when equations of motion unknowns are considered in error propagation and for generating the (approximate) closed-form transition matrix.

A discussion of the method of interpolation follows:

Method

Let

$$y_n = y(t_n)$$

$$\dot{y}_n = \dot{y}(t_n) \qquad (1)$$

be known at a sequence of values of t. An interpolation polynomial of degree s, $\bar{y}(t)$, may be determined for the interval (t_1, t_2) such that

$$\bar{y}(t_1) = y_1, \qquad \bar{y}(t_2) = y_2$$

$$\dot{\bar{y}}(t_1) = \dot{y}_1, \qquad \dot{\bar{y}}(t_2) = \dot{y}_2 \qquad (2)$$

We set

$$u = (t - t_1) \qquad (3)$$

and note that the conditions at t_1 are satisfied by

$$\bar{y} = y_1 + \dot{y}_1 u + \tfrac{1}{2} \ddot{y}_1 u^2 + \tfrac{1}{6} \dddot{y}_1 u^3 \qquad (4)$$

for any $\ddot{y}_1$, $\dddot{y}_1$. At $t = t_2$, $(u = t_2 - t_1)$,

$$\bar{y}(t_2) = y_2 = y_1 + h\dot{y}_1 + \tfrac{1}{2}h^2\ddot{y}_1 + \tfrac{1}{6}h^3\dddot{y}_1$$

$$\dot{\bar{y}}(t_2) = \dot{y}_2 = \dot{y}_1 + h\ddot{y}_1 + \tfrac{1}{2}h^2\dddot{y}_1 \qquad (5)$$

$$h = t_2 - t_1$$

and these equations may be solved for

$$h^2\ddot{y} = 6(y_2-y_1)-2h(\dot{y}_2 + 2\dot{y}_1)$$
$$h^3\dddot{y} = -12(y_2-y_1) + 6h(\dot{y}_2 + \dot{y}_1) \tag{6}$$

Then

$$\bar{y} = y_1 + u\dot{y}_1 + \frac{1}{2} u^2\ddot{y}_1 + \frac{1}{6} u^3\dddot{y}_1$$
$$\dot{\bar{y}} = \dot{y}_1 + u\ddot{y}_1 + \frac{1}{2} u^2\dddot{y}_1$$
$$\ddot{y}_1 = \frac{6}{h^2} (y_2-y_1) - \frac{2}{h}(\dot{y}_2 + 2\dot{y}_1) \tag{7}$$
$$\dddot{y}_1 = - \frac{12}{h^3} (y_2-y_1) + \frac{6}{h^2} (\dot{y}_2 + \dot{y}_1)$$

Let us now consider the selection of the spacing, h, for a given function, y(t). We will normally wish to minimize the number of intervals required for a given period, consistent with stated requirements on the accuracy of the interpolated values.

Let $y_3 = y(t_3)$ be a known point, with $t_3 \in (t_1,t_2)$. We may fit a polynomial of degree 4 through y_3 as well as y_1, y_2, $\dot{y}_1$, $\dot{y}_2$ by taking

$$\tilde{y} = \bar{y} + u^2(h-u)^2 f \tag{8}$$

where f is so chosen that $\tilde{y}(t_3) = y_3$.

Let

$$v = t_3-t_1$$
$$\epsilon = y_3-\bar{y}(v) \tag{9}$$

We have immediately

$$f = \epsilon / v^2 (h-v)^2$$

$$\tilde{y} = \bar{y} + \left(\frac{u(h-u)}{v(h-v)} \right)^2 \epsilon \tag{10}$$

The maximum of the absolute value of the error $\tilde{y}-\bar{y}$ occurs at $u = \frac{1}{2} h$ where

$$\max (\tilde{y}-\bar{y}) = \frac{h^4}{16} \frac{y_3 - \bar{y}(v)}{v^2 (h-v)^2} \tag{11}$$

If we now impose the condition

$$\max \frac{(\tilde{y}-\bar{y})}{\bar{y}} \leq M \tag{12}$$

as the criterion for selection of the spacing of tabulated coefficients, we may use the following selection process:

1. Let t_1 be the end-point of the previous interval.
2. Compute and save the interpolation coefficients for (t_1, t_2).
3. For each t_n in turn, $n = 3,4,\ldots$, compute the interpolation coefficients for (t_1, t_n), and compute the maximum error from (11).
4. If the test (12) is satisfied for all tested variables, replace the previously saved coefficients and return to step 2 for the next t_n. If not, store the coefficients for (t_1, t_{n-1}) and return to step 1.

1.3 PATCHED CONIC ERROR PROPAGATION

An error propagation program, PCEPP, using a "patched-conic" scheme for obtaining its trajectory, has been written in the process of checking subroutines for the Advanced Error Propagation Program (AEPP). This program has most of the capability to be delivered in the AEPP with the exception of course of a precision trajectory and consideration of equation of motion unknowns. Extensive checks have been made of this program against a similar patched-conic error propagation program developed by Philco for MSFC under Contract NAS8-11198. The latter program had been checked against the integrated-trajectory error propagation program provided to GSFC by Philco under Contract NAS5-3342 and had proven to give comparable (i.e good) results while exhibiting a better than 2-to-1 time advantage. The PCEPP runs two to five times faster than the patched conic program for MSFC on the Earth-to-Mars case being used for checkout.

Program options which have been checked out are; station measurement processing of range, range rate, azimuth and elevation; onboard optical measurement processing; station and beacon acquisition logic; and input (see Second Quarterly Progress Report). The PCEPP, estimated to be better than 90% checked out, still requires checkout of beacon measurement logic, onboard radar height and height-rate measurement logic, and inclusion of the speed of light uncertainty and the remaining station measurements.

Perhaps the most significant improvement in PCEPP over earlier programs is in the determination of acquisition (or contact) times for stations and beacons. This determination allows the program, when processing station or beacon measurement data, to move directly in time to acquisition, thus avoiding the many small steps previously required while testing for station vehicle visibility during the processing operation. The acquisition times for all stations and beacons to be considered are computed before processing and stored as critical events. The program can then proceed to the first station or beacon "on" time, keying in the station or beacon until its "off" time occurs. The AEPP, working from binary tape, will function in the same way.

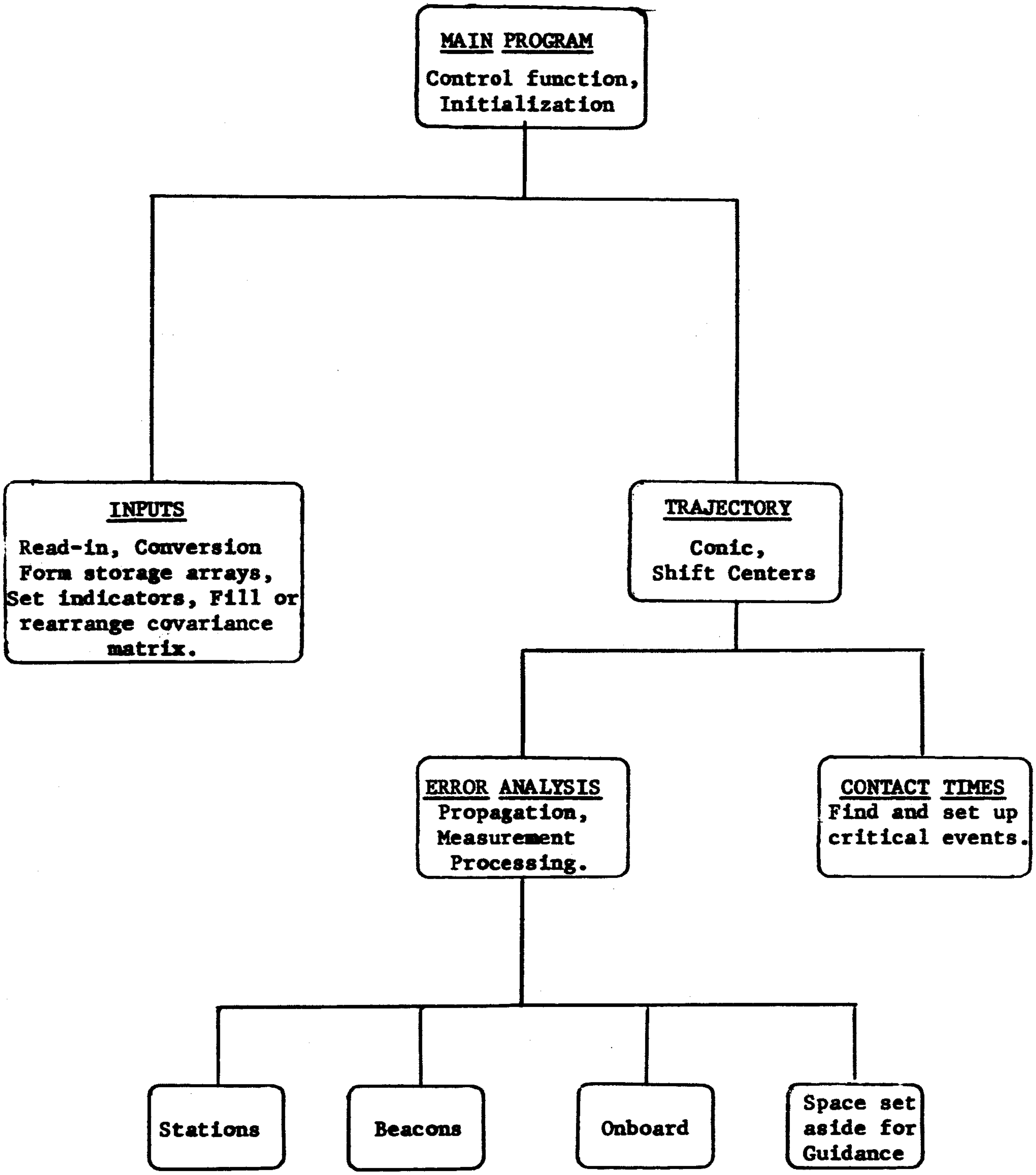
The program operates with a critical event philosophy, as indicated above, stopping at these events to consider and key in appropriate changes or to output certain information before proceeding toward the next event. The types of critical event are listed below.

1. regular output point
2. Earth-based tracking station acquisition or occultation
3. beacon acquisition or occultation
4. on-board optical measurement
5. special event time (anticipating guidance corrections, etc.)
6. time of patching to a new central body
7. stopping control time
8. final stopping time

The input and rearrangement of the expanded covariance matrix has been changed somewhat in the checkout process from the descriptions in the second and third quarterly reports for this contract. The space-saving gained by excluding some correlation terms for solved-for deterministic unknowns was outweighed by its handling complexity. The working covariance matrix is now (see the Third Quarterly Progress Report for old form)

	6 cols.	k cols.	L cols.	M cols.	N cols.
6 rows	P_{xx}	P_{xu_1}	P_{xv_1}	C_{xu}	C_{xv}
k rows	P_{xu_1}	$P_{u_1u_1}$	$P_{u_1v_1}$	C_{u_1u}	C_{u_1v}
L rows	P_{xv_1}	$P_{u_1v_1}$	$P_{v_1v_1}$	C_{v_1u}	C_{v_1v}

The many options and great flexibility of the AEPP have already required that the PCEPP should employ the OVERLAY feature of the IBSYS monitor because of storage size. The current form of the program is shown in the following diagram.


MAIN PROGRAM
Control function, Initialization
INPUTS
Read-in, Conversion Form storage arrays, Set indicators, Fill or rearrange covariance matrix.
TRAJECTORY
Conic, Shift Centers
ERROR ANALYSIS
Propagation, Measurement Processing.
CONTACT TIMES
Find and set up critical events.
Stations
Beacons
Onboard
Space set aside for Guidance

1.3.1 Sample Output

The figures which follow show sample output from the PCEPP and give an indication of current program capability. Another sample was shown in Figure 1 of the Third Quarterly Report concerning station and beacon parameters in effect.

Figure 1.3.1 shows first the date and state as input to the program. The starting epoch is February 10, 1965 at 1 minute, 25.15 seconds past 2 A.M. Injection conditions are input as Cartesian components of position and velocity in the equator and equinox of 1950 coordinate system, earth-centered. The Cartesian, spherical and orbital conditions (see Reference 1 for definition) follow, referred to equator and equinox of 1950 coordinates. The flight time to the target, Mars, is 300 days (although program stops at closest approach) and the starting time from epoch is zero.

The covariance matrix of estimation errors (P-matrix) follows as it was input. The input types are:

Type 0 Equator and equinox of 1950, Cartesian
Type 1 Local tangent plane, Cartesian
Type 2 Local Darboux coordinates, Cartesian
Type 3 Local tangent plane for positions, Darboux for velocity
Type 4 Altitude, down range, cross-range, velocity, path angle cross-range rate

Because the P-matrix is symmetrical only the upper half is output to facilitate locating particular elements. Position elements are given in kilometers, velocity elements in kilometers/second.

Dates, times from epoch and states are next given for conic patching to the Sun from the Earth and from Sun-orbit to Martian orbit.

```
STARTING CONDITIONS FOR CASE  1 ONBORD AND GUIDANCE CHECK 8/10/65

EARTH CENTERED    DATE 7502.10  FDATE  201.2515
                                                        INPUT TYPE CARTESIAN    EQUATOR      1950
   X-0.51940522E 04     Y-0.33714096E 04     Z-0.21758862E 04   XD 0.97623318E 01   YD-0.11540528E 02   ZD-0.54222573E 01
COORDINATES IN 1950, EQUATOR
   X-0.51940522E 04     Y-0.33714096E 04     Z-0.21758862E 04   DX 0.97623318E 01   DY-0.11540528E 02   DZ-0.54222573E 01
   R 0.65634641E 04   DEC-0.19360750E 02    RA-0.14701279E 03    V 0.16058885E 02  PTH-0.                 AZ 0.11097047E 03
 SMA-0.29217420E 04   ECC 0.32464214E 01   INC 0.28241553E 02  LAN 0.35212880E 03  APF 0.22447484E 03  RCA 0.65634642E 04
  C3 0.13642655E 03  THET 0.98911705E-02  PERV 0.16058885E 02  SLR 0.27871235E 05 IMPV 0.82658997E 01 TPER 0.81665611E-06

MARS   IS THE TARGET BODY
FLIGHT TIME   30000.0000DH.MS
REF. TIME         0.     DH.MS

                                       P MATRIX INPUT. TYPE 0
     0.91423638E 02    -0.61654378E 02    -0.26131453E 02    0.12808248E-00    0.31584869E-01    0.26233120E-01
                        0.50110430E 02     0.21750709E 02   -0.93802249E-01   -0.28877039E-01   -0.22218909E-01
                                           0.10357957E 02   -0.40481449E-01   -0.12995660E-01   -0.99779769E-02
                                                             0.20451091E-03    0.61378009E-04    0.41699710E-04
                                                                               0.39673257E-04   -0.17783660E-04
                                                                                                 0.75187547E-04

  FEB 10, 1975, 23 HRS, 43 MIN, 24.828 SEC                                               JULIAN DATE  2442454.48848181

CONDITIONS AT PATCH FROM EARTH  TO SUN                                      0 DAY 21 HRS 41 MIN 59.683 SEC
EARTH  CENTERED                                                                                      EQUATOR 1950
   X 0.75531931E 06     Y-0.49243303E 06     Z-0.20644846E 06   DX 0.96326062E 01   DY-0.61570763E 01   DZ-0.25674327E 01
   R 0.92499649E 06   DEC-0.12896371E 02    RA-0.33102477E 02    V 0.11717013E 02  PTH 0.89442758E 02   AZ 0.64656444E 02
 SMA-0.29217420E 04   ECC 0.32464180E 01   INC 0.28241556E 02  LAN 0.35212879E 03  APF 0.22447482E 03  RCA 0.65634540E 04
  C3 0.13642655E 03  THET 0.10738257E 03  PERV 0.16058890E 02  SLR 0.27871170E 05 IMPV 0.82658995E 01 TPER 0.90416204E 00
SUN    CENTERED                                                                                      EQUATOR 1950
   X-0.11441701E 09     Y 0.84261962E 08     Z 0.36545252E 08   DX-0.95000362E 01   DY-0.27582248E 02   DZ-0.11858094E 02
   R 0.14672043E 09   DEC 0.14423128E 02    RA 0.14363036E 03    V 0.31490403E 02  PTH-0.21196233E 02   AZ 0.10849929E 03
 SMA 0.16235350E 09   ECC 0.37254226E-00   INC 0.23301093E 02  LAN 0.29636771E-00  APF 0.23822470E 03  RCA 0.10186996E 09
  C3-0.81744741E 03  THET-0.97251723E 02  PERV 0.42286383E 02  SLR 0.13982082E 09 IMPV 0.61921615E 01 TPER-0.62402568E 02
  OCT  1, 1975, 21 HRS, 37 MIN, 35.275 SEC                                               JULIAN DATE  2442687.40110272

CONDITIONS AT PATCH FROM SUN    TO MARS                                   233 DAY 19 HRS 36 MIN 10.250 SEC
SUN    CENTERED                                                                                      EQUATOR 1950
   X 0.15990594E 09     Y 0.13624713E 09     Z 0.58323351E 08   DX-0.11156745E 02   DY 0.15235802E 02   DZ 0.65866848E 01
   R 0.21802478E 09   DEC 0.15516025E 02    RA 0.40432505E 02    V 0.19999676E 02  PTH 0.89180735E 01   AZ 0.72397356E 02
 SMA 0.16235348E 09   ECC 0.37254230E-00   INC 0.23301093E 02  LAN 0.29636747E-00  APF 0.23822469E 03  RCA 0.10186994E 09
  C3-0.81744750E 03  THET 0.16432824E 03  PERV 0.42286386E 02  SLR 0.13982081E 09 IMPV 0.61921625E 01 TPER 0.17051001E 03
MARS   CENTERED                                                                                      EQUATOR 1950
   X-0.44807599E 06     Y 0.27194399E 06     Z 0.20953199E 06   DX 0.43146401E 01   DY-0.26727889E 01   DZ-0.20488162E 01
   R 0.56447257E 06   DEC 0.21789618E 02    RA 0.14874581E 03    V 0.54733507E 01  PTH-0.89477977E 02   AZ 0.11179579E 03
 SMA-0.14419519E 04   ECC 0.37133292E 01   INC 0.30438476E 02  LAN 0.11615154E 02  APF 0.23798595E 03  RCA 0.39124890E 04
  C3 0.29805292E 02  THET-0.10509987E 03  PERV 0.71954732E 01  SLR 0.18440845E 05 IMPV 0.38311450E 01 TPER-0.11833233E 01
```

Figure 1.3.1

Figure 1.3.2 shows the miss-vector at closest approach to Mars (see subroutine BVE of Reference 1 for definitions)

B·T	ecliptic component of asymptotic miss
B·R	polar component of asymptotic miss
THE	true anomaly
VIN	velocity at infinite distance on target conic
RP	radius of closest approach
INC	orbital inclination
THD	true anomaly rate

Cartesian, spherical and orbital conditions at the end point are shown.

The next block of data shows sensitivities of B·T, B·R, time of flight, v-infinity, closest approach and inclination, to state (Cartesian, equator and equinox of 1950) deviations at the end point. These sensitivities are computed by differences, except for the sensitivity of v-infinity which is computed analytically from an incorrect equation for the run shown. The next block gives sensitivities of the same functions to state deviations at patch from Earth to Sun orbit. These sensitivities are computed using the transition matrix (the AO-matrix which is printed out next) and the sensitivities just described.

$$\frac{\partial(\text{miss functions})}{\partial(\text{state at Sun-patch})} = \frac{\partial(\text{miss functions})}{\partial(\text{state at end})}\,\frac{\partial(\text{state at end})}{\partial(\text{state at Sun-patch})}$$

Figure 1.3.3 shows output for optical onboard tracking, which is to be in effect from epoch to 1 day and 1 minute from epoch, with regular output each 12 hours. Measurements are to be made every 15 minutes (.01041666 days), with no time bias and relative to a reference plane for which the unit normal has a right ascension of 0° and a declination of 90° relative to Earth's equator. The measurements are to have random errors in K_1 and K_2 of the error model

```
                    MARS   CENTERED MISS VECTOR AT  235 DAY  0 HRS  0 MIN  9.250 SEC
 OCT  3, 1975,  2 HRS, 1 MIN, 34.209 SEC                                           JULIAN DATE  2442688.58442372
B.T-0.515491E 04 B.R-0.132807E 03 THE-0.850435E-06 VIN 0.545943E 01 RP  0.391249E 04 INC 0.120275E-00 THD 0.183911E-02
MARS  CENTERED COORDINATES, EQUATOR 1950
 X-0.14557734E 04    Y-0.32192578E 04    Z-0.16806738E 04   DX 0.66383895E 01   DY-0.19930353E 01   DZ-0.19324856E 01
 R 0.39124878E 04  DEC-0.25440054E 02   RA-0.11433285E 03    V 0.71954781E 01  PTH-0.38146973E-04   AZ 0.10730197E 03
SMA-0.14419489E 04  ECC 0.37133332E 01  INC 0.30438510E 02  LAN 0.11615154E 02  APF 0.23798597E 03  RCA 0.39124878E 04
 C3 0.29805356E 02 THET-0.              PERV 0.71954781E 01  SLR 0.18440859E 05 IMPV 0.38811494E 01 TPER-0.
```

PARTIALS OF TARGET VECTOR WRT STATE AT ENDPOINT

	X	Y	Z	XD	YD	ZD
B.T	0.30493617E-00	0.69984793E 00	0.32901624E-00	0.48925482E 03	-0.14648942E 03	-0.13632791E 03
B.R	0.19659890E-00	-0.57875062E 00	0.11165788E 01	-0.10225677E 03	0.19161995E 02	-0.19826710E 03
TIM	-0.16262469E-00	0.48857937E-01	0.47373944E-01	0.35768586E 02	0.78921656E 02	0.41201427E 02
VIN	-0.31174948E-03	-0.31766943E-03	-0.14328092E-03	0.48286819E 01	-0.35637695E 01	-0.26049384E 01
RP	-0.37212672E-00	-0.82292631E 00	-0.42908159E-00	0.16595366E-00	0.22539054E-00	0.94339138E-01
INC	-0.12180062E-04	0.59059387E-04	-0.10248907E-03	-0.12511268E-01	0.60875358E-01	-0.10576659E-00

PARTIALS OF TARGET VECTOR WRT STATE AT 0 DAY 21 HRS 41 MIN 59.683 SEC

	X	Y	Z	XD	YD	ZD
B.T	0.24230948E-00	0.13479895E 01	0.57544277E 00	0.53148346E 07	-0.63982259E 07	-0.28108955E 07
B.R	-0.14413472E 01	0.10517679E 01	-0.38126022E-00	-0.50154277E 07	-0.47209205E 07	-0.99738216E 07
TIM	0.17167617E 01	-0.96303714E 00	-0.44028619E-00	0.57204875E 07	0.95646635E 07	0.39006895E 07
VIN	0.12877142E-01	-0.56868348E-02	-0.26613887E-02	0.47553998E 05	0.63819706E 05	0.25638737E 05
RP	-0.20298351E-00	-0.13205301E 01	-0.54341715E 00	-0.50102400E 07	0.62472025E 07	0.29378827E 07
INC	0.59702710E-04	-0.41833451E-04	0.15970782E-04	0.21238305E 03	0.18944251E 03	0.40534583E 03

AD MATRIX

-0.11359268E 02	0.71946272E 01	0.31909537E 01	-0.35436274E 08	-0.67175563E 08	-0.28216511E 08
0.40905924E 01	-0.12706607E 01	-0.31520167E-00	0.17067745E 08	0.15869767E 08	0.90869334E 07
0.24761476E 01	-0.72287936E 00	-0.89543144E 00	0.96662902E 07	0.13243407E 08	0.18833565E 06
-0.16117013E-02	0.16151731E-02	0.74102126E-03	-0.31858289E 04	-0.12810514E 05	-0.51132999E 04
-0.41291309E-02	0.26465187E-02	0.10369862E-02	-0.12937751E 05	-0.23851591E 05	-0.11306461E 05
-0.15575259E-02	0.89012185E-03	0.68240184E-03	-0.48330675E 04	-0.10239378E 05	-0.15941861E 04

Figure 1.3.2

$$\sigma_\epsilon^2 = K_1^2 + K_2^2 \sin^{-1}\left(\frac{2\cdot\text{body radius}}{\text{range to body}}\right)$$

(this error model is erroneously described in the figure). The measurements to be made are right ascension and declination (type 2 angles) of the earth and moon relative to the reference plane (which for this case is the equatorial plane). The measurement cycle is

25 right ascension measurements of the earth, followed by
25 declination measurements of the earth, followed by
72 right-ascension measurements of the moon, followed by
72 declination measurements of the moon, at which

time and the cycle begins again. The standard deviations on K_1 and k_2 for each angle are 10 arc seconds and .000656%, respectively.

The statement that the covariance matrix is dimensioned 6 x 6 means that no deterministic unknowns are to be considered or solved for. The P-matrix is next printed out in Darboux coordinates. This matrix is normalized by dividing each row and column by the square root of the diagonal of that row and column. Rather than print the 1's along the diagonal, the square root of each diagonal element is printed. Units are still kilometers for positions and kilometers/second for velocities.

The case number, record number and event number appear before each output block. The event number tells the reason for the output, which in this case is an onboard optical measurement occuring. The record number tells which regular output interval we are in and the case number's meaning is obvious.

RMSP is the root-mean-square position estimate error, computed by square-rooting the first three diagonal elements of the P-matrix. RMSV is computed by square rooting the second three diagonal elements of the P-matrix.

```
NEW CONTROL TIMES AT     0 DAY  0 HRS  0 MIN  0.000 SEC
 START      0 DAY  0 HRS  0 MIN  0.000 SEC
  STOP      1 DAY  0 HRS  1 MIN  0.000 SEC
 DINTV      0 DAY 12 HRS  0 MIN  0.000 SEC
```

OPTICAL ONBOARD TRACKING IN EFFECT

```
PERIOD  0.10416666E-01 DAYS
 IBIAS  0.             SEC
REFRA.  0.             DEG
REFDEC  0.90000001E 02 DEG
```

RANDOM ERROR MODEL = (K1)**2+(K2**2)*(2.*RADIUS BODY/RANGE TO BODY)**2

BODY	NO. OF MEAS.	TYPE	K1 SEC	K2(PER-CENT)
EARTH	25	2	0.09999999E 02	0.65599999E-03
EARTH	25	2	0.09999999E 02	0.65599999E-03
MOON	72	2	0.09999999E 02	0.65599999E-03
MOON	72	2	0.09999999E 02	0.65599999E-03

THE COVARIANCE MATRIX IS DIMENSIONED 6 X 6

P MATRIX. STD DEV. ON DIAGONAL

	N	V	W	ND	VD	WD
N	0.38969979E 01	-0.72386514E 00	-0.54560811E-07	0.64194781E 00	-0.91503429E 00	0.38034680E-07
V		0.11661825E 02	0.17016931E-06	-0.94260614E 00	0.90155170E 00	0.15472967E-07
W			0.84098911E 00	-0.12938923E-05	-0.24827285E-05	-0.11276180E-01
ND				0.14578334E-01	-0.87573209E 00	-0.17267689E-07
VD					0.50260860E-02	-0.67615458E-07
WD						0.90322934E-02

ONBOARD OPTICAL MEAS OF RA-DEC TYPE OF EARTH

```
   0 DAY  0 HRS  0 MIN  0.000 SEC                                          CASE  1  REC.   1  EVENT 4
EARTH  CENTERED; EQUINOX OF 50
  X -0.51940522E 04   Y -0.33714096E 04   Z -0.21758862E 04  XD  0.97623318E 01  YD -0.11540528E 02  ZD -0.54222573E 01
                                     CURRENT RMS VALUES
RMSP= 0.8666388E 01 RMSV= 0.1441188E-01
```

RMS UNCERTAINTY IN MISS VECTOR AT MARS

```
B.T  0.15816472E 05 B.R  0.71674475E 05 TAR  0.66249348E 05 VIN  0.49300851E 03 RP   0.14777058E 05 INC  0.16892499E 03
```

Figure 1.3.3

Both refer to the post-observation P-matrix. The RMS Uncertainty in Miss Vector at Mars represents standard deviations in target miss functions obtained by propagating the P-matrix to the end point. TAR means time of arrival (seconds).

Figure 1.3.4 shows output of parametric data for three tracking stations, Goldstone, Johannesburg, and Woomera, which are being considered. Following this information the normalized P-matrix occurs, expressed in Darboux coordinates. The next six lines contain correlations between the six deterministic unknowns being solved for and the state (position and velocity). These correlations are initially zero. The deterministic unknowns for this case are station latitude and longitude for each of the three stations. The code for interpreting the unknowns (107, 108, 207, 208, 307, 308) is found in the Third Quarterly Report. The next block gives the correlations of these unknowns with themselves at initiation of the case.

Figure 1.3.5 lists the station on-off times for the run duration as well as range (km), azimuth (deg) and elevation (deg) at these critical events.

*** EARTH-BASED TRACKING IN EFFECT ***

NUMBER	NAME	LOCATION	OBSERVES
1	GUSTRC	LATITUDE = 35.38950 DEG LONGITUDE = 243.15176 DEG ALTITUDE =1037.53998 MET	AT INTERVALS OF1800.00 SEC WHEN ELEVATION IS ABOVE 0. DEG BUT LESS THAN 80.00 DEG

MEASUREMENT ERROR SOURCES	RANDOM	BIAS
RANGE RATE (MET/SEC)	0.03650	0.
AZIMUTH (MR)	0.03650	0.
ELEVATION (MR)	0.03650	0.
STATION LATITUDE (MET-N)	0.	100.0000
STATION LONG. (MET-EAST)	0.	100.0000

NUMBER	NAME	LOCATION	OBSERVES
2	JOHABG	LATITUDE = -25.88735 DEG LONGITUDE = 27.68478 DEG ALTITUDE =1391.91998 MET	AT INTERVALS OF1800.00 SEC WHEN ELEVATION IS ABOVE 0. DEG BUT LESS THAN 80.00 DEG

MEASUREMENT ERROR SOURCES	RANDOM	BIAS
RANGE RATE (MET/SEC)	0.03650	0.
AZIMUTH (MR)	0.03650	0.
ELEVATION (MR)	0.03650	0.
STATION LATITUDE (MET-N)	0.	100.0000
STATION LONG. (MET-EAST)	0.	100.0000

NUMBER	NAME	LOCATION	OBSERVES
3	WOMERA	LATITUDE = -31.39287 DEG LONGITUDE = 136.88502 DEG ALTITUDE = 15[illegible].79[illegible] MET	AT INTERVALS OF1800.00 SEC WHEN ELEVATION IS ABOVE 0. DEG BUT LESS THAN 80.00 DEG

MEASUREMENT ERROR SOURCES	RANDOM	BIAS
RANGE RATE (MET/SEC)	0.03650	0.
AZIMUTH (MR)	0.03650	0.
ELEVATION (MR)	0.03650	0.
STATION LATITUDE (MET-N)	0.	100.0000
STATION LONG. (MET-EAST)	0.	100.0000

P MATRIX. STD DEV. ON DIAGONAL

	N	V	W	ND	VD	WD
N	0.38969979E 01	-0.72386514E 00	-0.54560811E-07	0.64194781E 00	-0.91503429E 00	0.38034680E-07
V		0.11661825E 02	0.17016931E-06	-0.94260614E 00	0.90155170E 00	0.15472967E-07
W			0.84098911E 00	-0.12938923E-05	-0.24827285E-05	-0.11276180E-01
ND				0.14578334E-01	-0.87573209E 00	-0.17267689E-07
VD					0.50260860E-02	-0.67615458E-07
WD						0.90322934E-02
107	-0.	-0.	0.	-0.	-0.	0.
108	-0.	-0.	0.	-0.	-0.	0.
207	-0.	-0.	0.	-0.	-0.	0.
208	-0.	-0.	0.	-0.	-0.	0.
307	-0.	-0.	0.	-0.	-0.	0.
308	-0.	-0.	0.	-0.	-0.	0.

DETERMINISTIC ERRORS

	107	108	207	208	307	308
107	0.15678490E-04					
108	0.	0.19231891E-04				
207	0.	0.	0.15678490E-04			
208	0.	0.	0.	0.17427241E-04		
307	0.	0.	0.	0.	0.15678490E-04	
308	0.	0.	0.	0.	0.	0.18365199E-04

Figure 1.3.4

STATION CRITICAL EVENT AND CONDITIONS

TIME FROM EPOCH	EVENT	RANGE	AZIMUTH	ELEVATION
0 DAY 0 HRS 0 MIN 0.000 SEC	GOSTRC OFF	0.12665834E 05	52.340	-77.993
0 DAY 0 HRS 0 MIN 0.000 SEC	JOHABG OFF	0.17910144E 04	69.938	-1.997
0 DAY 0 HRS 0 MIN 0.000 SEC	WOMERA OFF	0.85599940E 04	-108.616	-40.411
0 DAY 0 HRS 6 MIN 15.951 SEC	WOMERA ON	0.54730304E 04	-98.289	0.017
0 DAY 2 HRS 8 MIN 19.177 SEC	JOHABG ON	0.96141825E 05	104.639	-0.087
0 DAY 7 HRS 44 MIN 34.185 SEC	WOMERA OFF	0.33519755E 06	-104.826	-0.000
0 DAY 12 HRS 53 MIN 45.062 SEC	GOSTRC ON	0.55330094E 06	106.357	-0.094
0 DAY 14 HRS 58 MIN 10.174 SEC	JOHABG OFF	0.64091557E 06	-104.066	0.000
0 DAY 18 HRS 48 MIN 23.250 SEC	WOMERA ON	0.80295064E 06	104.821	-0.078
0 DAY 23 HRS 33 MIN 44.339 SEC	GOSTRC OFF	0.10035220E 07	-106.053	0.092
1 DAY 2 HRS 10 MIN 53.868 SEC	JOHABG ON	0.11140360E 07	104.036	-0.028
1 DAY 7 HRS 47 MIN 23.995 SEC	WOMERA OFF	0.13506240E 07	-104.730	0.039
1 DAY 12 HRS 49 MIN 34.111 SEC	GOSTRC ON	0.15631059E 07	105.791	0.000
1 Day 14 HRS 56 MIN 57.884 SEC	JOHABG OFF	0.16527068E 07	-103.975	0.055
1 DAY 18 HRS 45 MIN 3.048 SEC	WOMERA ON	0.18131437E 07	104.785	-0.056
1 DAY 23 HRS 32 MIN 29.226 SEC	GOSTRC OFF	0.20153294E 07	-105.713	-0.000

Figure 1.3.5

SECTION 2

POWERED FLIGHT PROGRAMS

This section summarizes the current status of the Powered Flight Optimization and Error Analysis Programs.

2.1 PROGRAM STATUS

a. Trajectory Model

Description of equations describing the trajectory model and the techniques used for their solution.

Status: Completed. These equations and some techniques were reported in the First Quarterly Progree Report WDL-TR2332. The description of the remaining techniques have been prepared for inclusion in the final report.

b. Optimization Techniques

Derivation of equations describing the criteria for trajectory selection and the techniques used for their solution.

Status: Completed. The derivation of necessary conditions and of steepest descent solution of those conditions was reported in the Second Quarterly Progress Report, WDL-TR2407.

c. Error Sources

Description of error sources and their effect and the techniques used to determine the errors and/or their effect.

Status: Partially completed. The equations describing the effect of error sources and numerical techniques for trajectory and/or error determination were reported in WDL-TR2332 and WDL-TR2407, except for those error sources peculiar to powered flight. The model to be used for powered flight error sources is described in this report.

d. Input-Output

Description of data requirements, input-output options and formats.

Status: Nearly completed. Input formats and methods and certain output options were described in the Third Quarterly Progress Report WDL-TR2493. A major effort is being made to simplify the input formats and data requirements consistent with the maintenance of flexibility of program operation. The organization of output options for simple external control is the major remaining problem.

e. Programming

Status: Nearly completed. The programming of the Powered Flight Optimization Program is nearly completed, and the program is presently in checkout. The subroutines concerned with the trajectory simulation, including the equations of motion and adjoint equations, initial conditions, etc., have been completed, and many have been checked by independent check programs. The remaining programming effort includes checkout of the program as entity and the completion of a number of options for mission specification, program control, and output.

The Powered Flight Error Propagation Program is being programmed as a package of subroutines which modify the Interplanetary Error Propagation Program. To this end, tape formats have been selected for the storage of the vehicle ephemeris by both interplanetary and powered flight programs for use as input to the Error Propagation Program. A package of subroutines for storing and reading the ephemeris tape, tape positioning, etc., has been programmed and checked out, and the tape write logic has been included in the Optimization Program.

2.2 POWERED FLIGHT ERROR SOURCES

The definition of the model to be used for the inertial platform is given below. This model is believed adequate for all present and anticipated stabilized platform systems. This error model considers only the onboard measurement errors, and hence assumes perfect compensation by the guidance equations for power plant errors. Power plant errors may be included at a later date for any given steering and cutoff equations.

For convenience in describing the error sources and their effects, we define a coordinate system for each sensor and one for the platform itself. We take a right-handed cartesian coordinate frame, the PI-frame fixed at platform erection time with axes along the ideal or nominal directions of the gimbal axes. The transformation T_{I2PI} relating the PI-frame to the I-frame used in the trajectory simulation is a constant, dependent on the vehicle design.

We take a corresponding frame, the P-frame, fixed with respect to the platform so that the I - and P-frame axes coincide when the platform is perfectly erected. We let the G_i-frame be a right-handed coordinate frame with axes along the spin, input, and output axes, respectively, of the i^{th} platform stabilization gyro. The orientation of the G_i-frame relative to the P-frame depends only on the platform configuration, and is fixed for any given platform.

Finally, we let the A-frame axes lie along the nominal accelerometer axes. The orientation of the A-frame, too, is dependent only on platform configuration.

We consider the following error sources:

ϕ_{oi} = initial platform misalignment about the i^{th} axis,

$\dot{\phi}_i$ = steady drift of the i^{th} gyro,

μ_{Ii} = mass unbalance drift, input axis of the i^{th} gyro,

μ_{si} = mass unbalance drift, spin axis of the i^{th} gyro,

c_i = anisoelastic drift, i^{th} gyro,

α_i = bias, i^{th} accelerometer,

ϵ_i = scale factor error, i^{th} accelerometer,

s_i = threshold, i^{th} accelerometer,

β_{ij} = misalignment, i^{th} accelerometer into j^{th} axis, $i \neq j$.

Let t be the time from platform erection, and let $\bar{a}$ be the non-gravitational acceleration of the vehicle. We denote the total misalignment of the platform about the i^{th} axis by Φ_i (t). Using small angle assumptions, the transformation relating the I and P frame components of the acceleration,

$$a_P = T_{I2P}\, a_I$$

is

$$T_{I2P} = \begin{bmatrix} 1 & -\Phi_Z & \Phi_Y \\ \Phi_Z & 1 & -\Phi_X \\ -\Phi_Y & \Phi_X & 1 \end{bmatrix} \cdot T_{I2PI}$$

The accelerations along the i^{th} gyro axes and the ideal accelerometer axes are

$$a_{Gi} = T_{P2G_i} \; a_p$$

$$a_A = T_{P2A} \; a_p$$

where T_{P2Gi}, T_{P2A} are constant transformations dependent on platform configuration.

The drift of the i^{th} gyro, then, is

$$\dot{\Phi}_i = \dot{\phi}_i + \left[\mu_{si}, \mu_{Ii}, 0\right] \begin{bmatrix} a_{Gis} \\ a_{GiI} \\ a_{Gio} \end{bmatrix} + c_i \, a_{Gis}^2$$

Let

$$a_s = \begin{bmatrix} a_{sx} \\ a_{sy} \\ a_{sz} \end{bmatrix}$$

where a_{si} denotes the acceleration indicated by the i^{th} accelerometer. The true acceleration projected upon the x-accelerometer axis is

$$\begin{bmatrix} 0, & \beta_{XY}, & \beta_{XZ} \end{bmatrix} \begin{bmatrix} a_{AX} \\ a_{AY} \\ a_{AZ} \end{bmatrix}$$

etc., and hence the accelerations a_{si} are, for $\beta_{ii} = 0$

$$a_{si} = (a_i + s_i) + (1+\epsilon_i)a_{Ai} + \begin{bmatrix} \beta_{ix}, \beta_{iy}, B_{iz} \end{bmatrix} \begin{bmatrix} a_{AX} \\ a_{AY} \\ a_{AZ} \end{bmatrix}$$

Writing

$$B = \begin{bmatrix} \epsilon_x & \beta_{xy} & \beta_{xz} \\ \beta_{yx} & \epsilon_y & \beta_{yz} \\ \beta_{zx} & \beta_{zy} & \epsilon_z \end{bmatrix}$$

$$a_s = \begin{bmatrix} \alpha_x + s_x \\ \alpha_y + s_y \\ \alpha_z + s_z \end{bmatrix} + \left\{ I + B \right\} a_A$$

The platform orientation is obtained by integrating the drift rates, $\dot{\phi}_i$. Again neglecting second order terms in the errors

$$\phi_i = \dot{\phi}_i t + \begin{bmatrix} \mu_{si}, & \mu_{Ii}, & 0 \end{bmatrix} T_{P2Gi} \int_0^t a_{PI}(\tau)d\tau + c_i \int_0^t (q_i \; a_{PI}(\tau))^2 d\tau$$

q_i = first row of T_{P2Gi}

and the indicated acceleration is

$$a_s = \begin{bmatrix} \alpha_x + s_x \\ \alpha_y + s_y \\ \alpha_z + s_z \end{bmatrix} + \left\{ I + B \right\} \; T_{P2A} \; a_{PI} + T_{P2A} \; (\Phi \times a_{PI})$$

$$\Phi = \begin{bmatrix} \Phi_x \\ \Phi_y \\ \Phi_z \end{bmatrix}$$

APPENDIX A

SENSITIVITIES OF MEASUREMENTS TO STATE DEVIATIONS

A.1 MEASUREMENTS FROM EARTH-BASED TRACKING STATIONS

The measurements, y_i, to be considered are:

y_1	Range	distance measured from the tracker to the vehicle - magnitude of the slant range vector, which is directed toward the vehicle from the tracker.
y_2	Azimuth	angle from north at the tracker to the projection of the slant range vector into the tracker's local tangent plane - measured positive clockwise from north.
y_3	Elevation	angle between the slant range vector and the tracker's local tangent plane - positive up.
y_4	Right ascension	angle between the vernal equinox and the projection of the slant range vector into the equatorial plane - positive eastward from the equinox.
y_5	Declination	angle between the slant range vector and the equatorial plane - positive northward.
y_6	l - direction cosine	cosine of the angle between the slant range vector and a local north vector at the tracker.
y_7	m - direction cosine	cosine of the angle between the slant range vector and a local east vector at the tracker.

$\dot{y}_1$	Range rate
$\dot{y}_2$	Azimuth rate
$\dot{y}_3$	Elevation rate
$\dot{y}_4$	Right ascension rate
$\dot{y}_5$	Declination rate
$\dot{y}_6$	l - direction cosine rate
$\dot{y}_7$	m - direction cosine rate

The quantities with respect to which measurement sensitivities are to be found are:

R	Position state	assumed, for these derivations, to be the vector from the center of the earth to the vehicle.
V	Velocity state	assumed, for these derivations, to be the inertial velocity of the vehicle relative to the earth's center.
L	Station location deviation	vector deviation of the tracker's position from its nominal position, due to surveying errors, earth model, etc.
τ	Time bias	error in the tracking station's clock.
δ	Speed of light uncertainty	assumed to affect only range measurements.

NOTATION:

The following notation has been adopted through this section.

1. Vectors are represented by upper case letters (capitals).
2. Scalar quantities are denoted by lower case letters (Latin or Greek).
3. The magnitude (or length) of a vector is denoted either by the lower case symbol for that vector or by the conventional straight brackets. (e.g., the magnitude of R is r or $|R|$).
4. The conventional (·) and (x) are used for the vector dot product and cross product, respectively.
5. The symbol (I) denotes the identity matrix.
6. The symbol (Ax) denotes the skew-symmetric matrix:

$$\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

7. The symbol AA. denotes the symmetric matrix:

$$\begin{bmatrix} a_1a_1 & a_1a_2 & a_1a_3 \\ a_2a_1 & a_2a_2 & a_2a_3 \\ a_3a_1 & a_3a_2 & a_3a_3 \end{bmatrix}$$

8. The total time derivative of a quantity is denoted by a dot over that quantity.

Definitions. The following symbols will be used throughout this section for the stated quantities. Unless otherwise stated, all vectors are referred to the earth's equator and equinox of date coordinate system. The partial derivatives are thus in equator and equinox coordinates and must be transformed to equator and equinox of 1950.0 coordinates before being used in the error propagation program.

Symbol	Meaning
R	Vehicle position vector from the center of the earth.
V	Vehicle velocity vector from the center of the earth (note that $V = \dot{R}$).
R_T	Tracker position vector from the center of the earth.
S	Vehicle position vector from the tracker - the slant range vector (note that $S = R - R_T$).
N	Unit north vector at the tracker.
E	Unit east vector at the tracker.
D	Unit down (local vertical) vector at the tracker. (Note that D is not generally parallel to R_T unless the earth is assumed spherical).
Ω	Earth's sidereal angular velocity vector - assumed parallel to the north polar axis.
I_e	Unit vector along the vernal equinox of date.
J_e	Unit vector in the equator of date 90° east of I_e.
K_e	Unit vector along the north polar axis of date.
ϵ_n	Eastward component of station location error.

ϵ_e Eastward component of station location error.

ϵ_d Station altitude error (positive down).

R_T^o Nominal tracker position vector (note that $R_T = R_T^o + \epsilon_n N + \epsilon_e E + \epsilon_d D$).

Δ Station location deviation vector expressed in the N, E, D coordinate system. $(\Delta = (\epsilon_n, \epsilon_e, \epsilon_d))$.

Preparatory Observations. The following relationships for any vectors A, B and C are used in this section.

1. $AxB = -BxA$ (1)
2. $A \cdot BxC = AxB \cdot C = B \cdot CxA$ (2)
3. $AxA = 0$ (3)
4. $A \cdot A = a^2$ (4)
5. $Ax(BxC) = B(A \cdot C) - C(A \cdot B)$ (5)
6. $(Ax)(Ax) = (AA.) - a^2 I$ (6)
7. $\frac{d}{dt}(A \cdot B) = \dot{A} \cdot B + A \cdot \dot{B}$ (7)
8. $\frac{d}{dt}(AxB) = \dot{A}xB + Ax\dot{B}$ (8)
9. $\frac{d}{dt}\left(\frac{A}{a^n}\right) = \frac{1}{a^n}\left(I - \frac{nAA.}{a^2}\right)\dot{A}$ (9)

Time derivatives of R_T, E, N and D are easily shown to be

$$\begin{aligned} \dot{R}_T &= \Omega \times R_T \\ \dot{N} &= \Omega \times N \\ \dot{E} &= \Omega \times E \\ \dot{D} &= \Omega \times D \end{aligned} \tag{10}$$

Thus, since $S = R - R_T$, we have

$$\dot{S} = V - \Omega \times R_T$$
$$\ddot{S} = \dot{V} - \Omega \times \dot{R}_T = \dot{V} - \Omega \times (\Omega \times R_T) \tag{11}$$

The tracker position, R_T, is given in terms of the nominal tracker position, R_T^o, and the station location errors as follows.

$$R_T = R_T^o + \epsilon_n N + \epsilon_e E + \epsilon_d D$$

Thus, R_T has the following sensitivity to ϵ_n, ϵ_e and ϵ_d.

$$\frac{\partial R_T}{\partial \epsilon_n} = N$$

$$\frac{\partial R_T}{\partial \epsilon_e} = E \tag{12}$$

$$\frac{\partial R_T}{\partial \epsilon_d} = D$$

The directonal unit vectors at the tracker, E, N and D, are, in general, affected by station location errors. The sensitivities are derived as follows. We have, to first order

$$\delta(\text{Latitude}) = \frac{\epsilon_n}{|R_T|}$$

$$\delta(\text{Longitude}) = \frac{\epsilon_e}{|R_T| \cos(\text{Lat})}$$

$$\delta(\text{Altitude}) = -\epsilon_d$$

$$E = \begin{bmatrix} -\sin(Lon) \\ \cos(Lon) \\ 0 \end{bmatrix}, \quad N = \begin{bmatrix} -\sin(Lat)\cos(Lon) \\ -\sin(Lat)\sin(Lon) \\ \cos(Lat) \end{bmatrix}, \quad D = - \begin{bmatrix} \cos(Lat)\cos(Lon) \\ \cos(Lat)\sin(Lon) \\ \sin(Lat) \end{bmatrix}$$

$$\delta E = \frac{\partial E}{\partial Lat}\,\delta(Lat) + \frac{\partial E}{\partial Lon}\,\delta(Lon)$$

$$= \frac{\partial E}{\partial Lon}\,\partial(Lon)$$

$$= \begin{bmatrix} -\cos(Lon) \\ -\sin(Lon) \\ 0 \end{bmatrix} \delta(Lon)$$

$$= (N\sin(Lat) + D\cos(Lat))\,\frac{\epsilon_e}{|R_T|\cos(Lat)}$$

$$= \frac{\epsilon_e}{|R_T|}\left[N\tan(Lat) + D\right]$$

$$\delta N = \frac{\partial N}{\partial Lat}\,\delta(Lat) + \frac{\partial N}{\partial Lon}\,\delta(Lon)$$

$$= \begin{bmatrix} -\cos(Lat)\cos(Lon) \\ -\cos(Lat)\sin(Lon) \\ -\sin(Lat) \end{bmatrix} \delta(Lat) + \begin{bmatrix} \sin(Lat)\sin(Lon) \\ -\sin(Lat)\cos(Lon) \\ 0 \end{bmatrix} \delta(Lon)$$

$$= D\,\frac{\epsilon_n}{|R_T|} - E\,\sin\,(Lat)\,\frac{\epsilon_e}{|R_T|\,\cos(Lat)}$$

$$= \frac{1}{|R_T|}\left(\epsilon_n D - \epsilon_e E\tan(Lat)\right)$$

$$\delta D = \frac{\partial D}{\partial Lat}\,\delta(Lat) + \frac{\partial D}{\partial Lon}\,\delta(Lon)$$

$$= \begin{bmatrix} \sin(Lat)\cos(Lon) \\ \sin(Lat)\sin(Lon) \\ -\cos(Lat) \end{bmatrix} \delta(Lat) + \begin{bmatrix} \cos(Lat)\sin(Lon) \\ -\cos(Lat)\cos(Lon) \\ 0 \end{bmatrix} \delta(Lon)$$

$$= -N \frac{\epsilon_n}{|R_T|} - E\cos(Lat) \frac{\epsilon_e}{|R_T|\cos(Lat)}$$

$$= -\frac{1}{|R_T|}(\epsilon_n N + \epsilon_e E)$$

It follows simply from the above that

$$\frac{\partial E}{\partial \epsilon_e} = \frac{N\tan(Lat) + D}{|R_T|}, \quad \frac{\partial N}{\partial \epsilon_e} = -\frac{E\tan(Lat)}{|R_T|}, \quad \frac{\partial D}{\partial \epsilon_e} = \frac{-E}{|R_T|} \qquad (13)$$

$$\frac{\partial E}{\partial \epsilon_n} = 0, \quad \frac{\partial N}{\partial \epsilon_n} = +\frac{D}{|R_T|}, \quad \frac{\partial D}{\partial \epsilon_n} = \frac{-N}{|R_T|} \qquad (14)$$

$$\frac{\partial E}{\partial \epsilon_d} = 0, \quad \frac{\partial N}{\partial \epsilon_d} = 0, \quad \frac{\partial D}{\partial \epsilon_d} = 0 \qquad (15)$$

A.1.1 <u>Range, Y</u>

The range measurement is given by

$$y_1 = |S| = s \qquad (a)$$

where S is the slant range vector.

The position gradient, $\frac{\partial y_1}{\partial R}$, is found by using the chain rule and

$$\frac{\partial S}{\partial R} = I$$

$$\frac{\partial y_1}{\partial R} = \frac{\partial y_1}{\partial S} \frac{\partial S}{\partial R} = \frac{\partial y_1}{\partial S} = \frac{\partial (S \cdot S)^{\frac{1}{2}}}{\partial S} = \frac{S}{s} \tag{b}$$

The velocity gradient, $\frac{\partial y_1}{\partial V}$, is zero because V does not explicitly enter the formulation for y_1.

$$\frac{\partial y_1}{\partial V} = 0 \tag{c}$$

The partial derivatives of range with respect to station location errors are found from the chain rule and from $\frac{\partial S}{\partial R_T} = -I$

$$\frac{\partial y_1}{\partial \epsilon_e} = \frac{\partial y_1}{\partial S} \frac{\partial S}{\partial R_T} \frac{\partial R_T}{\partial \epsilon_e} = \left(\frac{S}{s} \right) (-I)\ (E) = -\left(\frac{S}{s} \right) \cdot E \tag{d}$$

Similarly,

$$\frac{\partial y_1}{\partial \epsilon_n} = \left(\frac{S}{s} \right) \cdot N \tag{d'}$$

$$\frac{\partial y_1}{\partial \epsilon_d} = -\left(\frac{S}{s} \right) \cdot D \tag{d''}$$

Combining the above three equations into a vector, we have

$$\frac{\partial y_1}{\partial \Delta} = -\frac{\partial y_1}{\partial R} \left(N\ E\ D \right)$$

The partial derivative of range with respect to time bias is taken to be the total time derivative of range

$$\frac{\partial y_1}{\partial_\tau} \equiv \dot{y}_1 = \frac{\partial y_1}{\partial S} \cdot \dot{S} = \frac{S}{s} \cdot \dot{S} \qquad \text{(e)}$$

The partial derivative of range with respect to the speed of light is given by

$$\frac{\partial y_1}{\partial(\delta c)} = 2\frac{s}{c} \qquad \text{(f)}$$

A.1.2 <u>Azimuth, y_2</u>

The azimuth measurement formulation is

$$y_2 = \tan^{-1}\left(\frac{S \cdot E}{S \cdot N}\right) \qquad \text{(a)}$$

The position gradient, $\frac{\partial y_2}{\partial R}$, is

$$\frac{\partial y_2}{\partial R} = \frac{\partial y_2}{\partial S} = \frac{1}{1+\left(\frac{S \cdot E}{S \cdot N}\right)^2} \frac{\partial}{\partial S}\left(\frac{S \cdot E}{S \cdot N}\right)$$

$$= \frac{S \cdot N^2}{S \cdot N^2 + S \cdot E^2}\left[\frac{(S \cdot N)E - (S \cdot E)N}{S \cdot N^2}\right]$$

and using equation (5) we may write

$$\frac{\partial y_2}{\partial R} = \frac{S \times (E \times N)}{S \cdot N^2 + S \cdot E^2} = \frac{D \times S}{|D \times S|^2} \qquad \text{(b)}$$

The velocity gradient of azimuth is zero.

$$\frac{\partial y_2}{\partial V} = 0 \tag{c}$$

The partial derivatives of azimuth with respect to station location errors are found by using the chain rule, the fact that

$$\frac{\partial y_2}{\partial R_T} = - \frac{\partial y_2}{\partial R}$$, and equations (13), (14), and (15).

$$\frac{\partial y_2}{\partial \epsilon_e} = - \frac{\partial y_2}{\partial R_T} \frac{\partial R_T}{\partial \epsilon_e} + \frac{\partial y_2}{\partial E} \frac{\partial E}{\partial \epsilon_e} + \frac{\partial y_2}{\partial N} \frac{\partial N}{\partial \epsilon_e}$$

$$\frac{\partial y_2}{\partial \epsilon_2} = - \frac{\partial y_2}{\partial R} E + \frac{(S \cdot N)S}{|D \times S|^2} \cdot \frac{(N\tan(Lat) + D)}{|R_T|} + \frac{(-S \cdot E)S}{|D \times S|^2} \cdot \left(\frac{-E\tan(Lat)}{|R_T|}\right)$$

$$= - \frac{\partial y_2}{\partial R} \cdot E + \frac{\tan(Lat)}{|R_T|} + \frac{(S \cdot N)(S \cdot D)}{|R_T|\ |D \times S|^2} \tag{d}$$

$$\frac{\partial y_2}{\partial \epsilon_n} = \frac{\partial y_2}{\partial R_T} \frac{\partial R_T}{\partial \epsilon_n} + \frac{\partial y_2}{\partial E} \frac{\partial E}{\partial \epsilon_n} + \frac{\partial y_2}{\partial N} \frac{\partial N}{\partial \epsilon_n}$$

$$= - \frac{\partial y_2}{\partial R} N + 0 + \frac{(-S \cdot E)S}{|D \times S|^2} \cdot \left(\frac{D}{|R_T|}\right)$$

$$= - \frac{\partial y_2}{\partial R} N - \frac{(S \cdot E)(S \cdot D)}{|R_T|\ |D \times S|^2} \tag{d'}$$

$$\frac{\partial y_2}{\partial \epsilon_d} = - \frac{\partial y_2}{\partial R} D = - \frac{(D \times S)}{|D \times S|^2} \cdot D = 0 \tag{d''}$$

Combining the above three equations, we have

$$\frac{\partial y_2}{\partial \Delta} = -\frac{\partial y_2}{\partial R}\left(N\ E\ D\right) + \left[\frac{-(S\cdot E)(S\cdot D)}{|R_T|\ |D \times S|^2},\ \frac{\tan(Lat)}{|R_T|} + \frac{(S\cdot N)(S\cdot D)}{|R_T|\,|D \times S|^2},\ 0\right]$$

The partial derivative of azimuth with respect to time bias is taken to be

$$\frac{\partial y_2}{\partial \tau} \equiv \dot{y}_2 = \frac{\partial y_2}{\partial S}\dot{S} + \frac{\partial y_2}{\partial E}\dot{E} + \frac{\partial y_2}{\partial N}\dot{N}$$

$$= \frac{1}{|D \times S|^2}\left\{(D \times S)\cdot \dot{S} + (S\cdot N)\ S\cdot\dot{E} - (S\cdot\dot{E})\cdot\dot{N}\right\}$$

$$= \frac{1}{|D \times S|^2}\left\{(D \times S)\cdot \dot{S} + (S\cdot N)\ S\cdot\Omega \times E - (S\cdot E)\ S\cdot\Omega \times N\right\}$$

$$= \frac{1}{|D \times S|^2}\left\{(D \times S)\cdot \dot{S} + (S \times \Omega)\cdot\left[(S\cdot N)\ E - (S\cdot E)N\right]\right\}$$

$$= \frac{(D \times S)}{|D \times S|^2}\left\{\dot{S} + S \times \Omega\right\}$$

$$= \frac{\partial y_2}{\partial R}\left\{\dot{S} + S \times \Omega\right\} \qquad \text{(e)}$$

A.1.3 Elevation, y_3

The formulation of elevation measurement from known quantities is

$$y_3 = \sin^{-1}\left(-\frac{S\cdot D}{s}\right) \qquad \text{(a)}$$

The position gradient, $\frac{\partial y_3}{\partial R}$, is found by using equation (6) and a variation of equation (9).

$$\frac{\partial y_3}{\partial R} = \frac{\partial y_3}{\partial S} = \frac{-sD\cdot}{|D \times S|}\left[\frac{I}{s} - \frac{SS\cdot}{s^3}\right]$$

$$= + \frac{D \cdot (Sx)(Sx)}{|D \times S| \ s^2}$$

$$= \frac{(D \times S) \times S}{s^2 |D \times S|} \qquad \text{(b)}$$

The velocity gradient of elevation is zero because elevation does not explicitly depend on velocity.

$$\frac{\partial y_3}{\partial V} = 0 \qquad \text{(c)}$$

The partial derivative of elevation with respect to station location errors is found using the chain rule.

$$\frac{\partial y_3}{\partial \epsilon_e} = \frac{\partial y_3}{\partial R_T}\frac{\partial R_T}{\partial \epsilon_e} + \frac{\partial y_3}{\partial D}\frac{\partial D}{\partial \epsilon_e}$$

$$= -\frac{\partial y_3}{\partial R} E + \frac{S\cdot}{|D \times S|}\left(\frac{E}{|R_T|}\right) \qquad \text{(d)}$$

$$\frac{\partial y_3}{\partial \epsilon_n} = \frac{\partial y_3}{\partial R_T}\frac{\partial R_T}{\partial \epsilon_n} + \frac{\partial y_3}{\partial D}\frac{\partial D}{\partial \epsilon_n} \qquad \text{(d')}$$

$$= -\frac{\partial y_3}{\partial R} \cdot N + \frac{S}{|D \times S|}\left(\frac{N}{|R_T|}\right)$$

$$\frac{\partial y_3}{\partial \epsilon_d} = \frac{\partial y_3}{\partial R_T}\frac{\partial R_T}{\partial \epsilon_d} + \frac{\partial y_3}{\partial D}\frac{\partial D}{\partial \epsilon_d}$$

$$= -\frac{\partial y_3}{\partial R} D + 0 \qquad \text{(d'')}$$

Combining the three above equations we have

$$\frac{\partial y_3}{\partial \Delta} = - \frac{\partial y_3}{\partial R} \left(N \; E \; D \right) + \frac{1}{|R_T| \; |D \times S|} (S \cdot N, \; S \cdot E, \; 0)$$

The partial derivative of elevation with respect to time bias is defined as the elevation rate.

$$\frac{\partial y_3}{\partial \tau} \equiv \dot{y}_3 = \frac{\partial y_3}{\partial S} \dot{S} + \frac{\partial y_3}{\partial U}$$

$$= \frac{\partial y_3}{\partial R} \dot{S} + \frac{S \cdot}{|S \times U|} (\Omega \times U)$$

$$= \frac{\partial y_3}{\partial R} (\dot{S} + S \times \Omega)$$

A.1.4 Right Ascension, y_4

The right ascension measurement may be formulated in terms of the slant range vector and the bias vectors of the equator and equinox of date coordinate system.

$$y_4 = \tan^{-1} \left(\frac{S \cdot J_e}{S \cdot I_e} \right) \tag{a}$$

The position gradient, $\frac{\partial y_4}{\partial R}$, is given by

$$\frac{\partial y_4}{\partial R} = \frac{\partial y_4}{\partial S} = \frac{S \cdot I_e^2}{|S \times Ke|^2} \left\{ \frac{(S \cdot I_e) J_e - (S \cdot J_e) I_e}{S \cdot I_e^2} \right\}$$

$$= \frac{1}{|S \times Ke|^2} \left\{ Sx(J_e \times I_e) \right\}$$

$$= \frac{(K_e \times S)}{|K_e \times S|^2} \qquad \text{(b)}$$

The velocity gradient is, of course, zero.

$$\frac{\partial y_4}{\partial V} = 0 \qquad \text{(c)}$$

The sensitivity of right ascension to station location errors is computed simply, since I_e, J_e and K_e are assumed independent of station location.

$$\frac{\partial y_4}{\partial \epsilon_e} = \frac{\partial y_4}{\partial R_T} \frac{\partial R_T}{\partial \epsilon_e} = \frac{-\partial y_4}{\partial R} E \qquad \text{(d)}$$

$$\frac{\partial y_4}{\partial \epsilon_n} = \frac{\partial y_4}{\partial R_T} \frac{\partial R_T}{\partial \epsilon_n} = \frac{-\partial y_4}{\partial R} N \qquad \text{(d')}$$

$$\frac{\partial y_4}{\partial \epsilon_d} = \frac{\partial y_4}{\partial R_T} \frac{\partial R_T}{\partial \epsilon_d} = \frac{-\partial y_4}{\partial R} D \qquad \text{(d'')}$$

Summarizing the above three equations,

$$\frac{\partial y_4}{\partial \Delta} = \frac{-\partial y_4}{\partial R} \left(N \; E \; D \right)$$

The partial of right ascension with respect to time bias is taken to be right ascension rate.

$$\frac{\partial y_4}{\partial_\tau} \equiv \dot{y}_4 = \frac{\partial y_4}{\partial S}\dot{S} = \frac{\partial y_4}{\partial R}\dot{S} \qquad \text{(e)}$$

A.1.5 <u>Declination, y_5</u>

The declination measurement is given by

$$y_5 = \sin^{-1}\left(\frac{S \cdot K_e}{s}\right) \qquad \text{(a)}$$

The position gradient, $\frac{\partial y_5}{\partial R}$, is

$$\frac{\partial y_5}{\partial R} = \frac{\partial y_5}{\partial S} = \frac{sK_e \cdot}{|K_e \times S|}\left[\frac{I}{s} - \frac{SS \cdot}{s^3}\right]$$

$$= \frac{-K_e \cdot (S_x)(S_x)}{s^2 |K_e \times S|}$$

$$= \frac{Sx(K_e \times S)}{s^2 |K_e \times S|} \qquad \text{(b)}$$

The velocity gradient is zero.

$$\frac{\partial y_5}{\partial V} = 0 \qquad \text{(c)}$$

As with right ascension, declination sensitivity to station location errors is simple in formulation.

$$\frac{\partial y_5}{\partial \epsilon_e} = \frac{\partial y_5}{\partial R_T}\frac{\partial R_T}{\partial \epsilon_e} = -\frac{\partial y_5}{\partial R} E \qquad \text{(d)}$$

$$\frac{\partial y_5}{\partial \epsilon_n} = \frac{\partial y_5}{\partial R_T}\frac{\partial R_T}{\partial \epsilon_n} = -\frac{\partial y_5}{\partial R} N \qquad \text{(d')}$$

$$\frac{\partial y_5}{\partial \epsilon_d} = \frac{\partial y_5}{\partial R_T}\frac{\partial R_T}{\partial \epsilon_d} = -\frac{\partial y_5}{\partial R} D \qquad \text{(d'')}$$

In summary,

$$\frac{\partial y_5}{\partial \Delta} = \frac{-\partial y_5}{\partial R}\left(N \; E \; D \right)$$

The partial derivative of declination with respect to time bias is defined to be declination rate.

$$\frac{\partial y_5}{\partial \tau} = \dot{y}_5 \; \frac{\partial y_5}{\partial S}\dot{S} = \frac{\partial y_5}{\partial R}\dot{S} \qquad \text{(e)}$$

A.1.6 1 Direction Cosine, y_6

This measurement is formulated by

$$y_6 = \frac{S \cdot N}{s} \qquad \text{(a)}$$

The position gradient, $\frac{\partial y_6}{\partial R}$, is

$$\frac{\partial y_6}{\partial R} = \frac{\partial y_6}{\partial S} = \frac{N\cdot}{s}\left[I - \frac{SS\cdot}{s^2} \right] \qquad \text{(b)}$$

$$= \frac{Sx(N \times S)}{s^3}$$

This measurement is not an explicit function of velocity.

$$\frac{\partial y_6}{\partial V} = 0 \tag{c}$$

Station location errors affect this measurement through R_T and N.

$$\frac{\partial y_6}{\partial \epsilon_e} = \frac{\partial y_6}{\partial R_T}\frac{\partial R_T}{\partial \epsilon_e} + \frac{\partial y_6}{\partial N}\frac{\partial N}{\partial \epsilon_e}$$

$$= \frac{-\partial y_6}{\partial R} E + \frac{S}{s}\cdot\left(\frac{-E\tan(Lat)}{|R_T|}\right) \tag{d}$$

$$\frac{\partial y_6}{\partial \epsilon_n} = \frac{\partial y_6}{\partial R_T}\frac{\partial R_T}{\partial \epsilon_n} + \frac{\partial y_6}{\partial N}\frac{\partial N}{\partial \epsilon_n}$$

$$= \frac{-\partial y_6}{\partial R} N + \frac{S}{s}\cdot\left(\frac{D}{|R_T|}\right) \tag{d'}$$

$$\frac{\partial y_6}{\partial \epsilon_d} = \frac{\partial y_6}{\partial R_T}\frac{\partial R_T}{\partial \epsilon_d} + \frac{\partial y_6}{\partial N}\frac{\partial N}{\partial \epsilon_d}$$

$$= \frac{-\partial y_6}{\partial R} D \tag{d''}$$

The above three equations may be summarized.

$$\frac{\partial y_6}{\partial \Delta} = \frac{-\partial y_6}{\partial R}\left(N\ E\ D\right) - \frac{1}{s|R_T|}\left(-S\cdot D,\ S\cdot E\tan(Lat),\ 0\right)$$

The partial derivative with respect to time bias is taken to be the measurement rate.

$$\frac{\partial y_6}{\partial \tau} \equiv \dot{y}_6 = \frac{\partial y_6}{\partial S}\dot{S} + \frac{\partial y_6}{\partial N}\dot{N} = \frac{\partial y_6}{\partial R}\dot{S} + \frac{S}{s}\cdot(\rho \times N) = \frac{\partial y_6}{\partial R}\left(\dot{S} + S\times\rho\right) \tag{e}$$

A.1.7 m-Direction Cosine, y_7

The derivations of partial derivatives for this measurement follow directly the derivations for the l-direction cosine (A.1.6), with the substitution of E for N.

$$y_7 = \frac{S \cdot E}{s} \qquad \text{(a)}$$

$$\frac{\partial y_7}{\partial R} = \frac{S x (E \times S)}{s^3} \qquad \text{(b)}$$

$$\frac{\partial y_7}{\partial V} = 0 \qquad \text{(c)}$$

$$\frac{\partial y_7}{\partial \Delta} = \frac{-\partial y_7}{\partial R} \begin{pmatrix} N & E & D \end{pmatrix} + \frac{S}{s \, |R_T|} (0, N\tan(Lat) + D, 0) \qquad \text{(d)}$$

$$\frac{\partial y_7}{\partial \tau} \equiv \dot{y}_7 = \frac{\partial y_7}{\partial R} (\dot{S} + S \times \Omega) \qquad \text{(e)}$$

A.1.8 Range Rate, $\dot{y}_1$

This measurement has been derived in (A.1.1), equation (e) to be

$$\dot{y}_1 = \frac{S}{s} \cdot \dot{S} \qquad \text{(a)}$$

and may also be seen to equal $\frac{\partial y_1}{\partial R} (\dot{S} + \Omega \times S)$.

The position gradient, $\frac{\partial \dot{y}_1}{\partial R}$, may be obtained either by carrying out the gradient of (a) formally or by noting that for any y,

$$\frac{\partial \dot{y}}{\partial R} = \frac{d}{dt} \left(\frac{\partial y}{\partial R} \right).$$

By either derivation, the result is

$$\frac{\partial \dot{y}_1}{\partial R} = \frac{S x(\dot{S} \times S)}{s^3} = \frac{1}{s}\left[\dot{S} - \frac{(S \cdot \dot{S})S}{s^2}\right] \quad \text{(b)}$$

The velocity gradient has already been derived, as may be seen from the fact that for any y,

$$\frac{\partial \dot{y}}{\partial V} = \frac{\partial y}{\partial R}$$

Therefore, borrowing a result from (A.1.1)

$$\frac{\partial \dot{y}_1}{\partial V} = \frac{\partial y_1}{\partial R} = \frac{S}{s} \quad \text{(c)}$$

Sensitivity of this measurement to station location errors is found by the interchange of order of differentiation described in (b) above and the results of (A.1.1) equation (d).

$$\frac{\partial \dot{y}_1}{\partial \Delta} = \frac{d}{dt}\left(\frac{\partial y_1}{\partial \Delta}\right) = -\frac{d}{dt}\left(\frac{\partial y_1}{\partial R}\right)\left(N\ E\ D\right) + \frac{\partial y_1}{\partial R}\left(\dot{N}\ \dot{E}\ \dot{D}\right)$$

$$= \left[-\frac{\partial \dot{y}_1}{\partial R} + \frac{\partial y_1}{\partial R} \cdot (\omega x)\right]\left(N\ E\ D\right) \quad \text{(d)}$$

The time bias partial derivative is taken to be the second derivative of range with respect to time, and equation (9) is used in its derivation.

$$\frac{\partial \dot{y}_1}{\partial \tau} \equiv \frac{d\dot{y}_1}{dt} = \frac{\partial \dot{y}_1}{\partial R} \cdot \dot{S} + \frac{\partial y_1}{\partial R} \cdot \ddot{S} \quad \text{(e)}$$

The symbol, $\ddot{S}$, has been defined in the second of equations (11) and needs to be computed only if time biases are considered simultaneously with measurement rates.

A.1.9 Azimuth rate, $\dot{y}_2$

This measurement is (from (e) of A.1.2)

$$\dot{y}_2 = \frac{\partial y_2}{\partial R} \cdot \left\{ \dot{S} + S \times \Omega \right\} = \frac{(D \times S)\cdot}{|D \times S|^2} \left\{ \dot{S} + S \times \Omega \right\} \tag{a}$$

The position gradient of azimuth rate, $\frac{\partial \dot{y}_2}{\partial R}$, is found by use of the chain rule and equation (9).

$$\frac{\partial \dot{y}_2}{\partial R} = \frac{\partial \dot{y}_2}{\partial S} = \frac{\partial \dot{y}_2}{\partial (D \times S)} \frac{\partial (D \times S)}{\partial S} + \frac{\partial \dot{y}_2}{\partial (S \times \Omega)} \frac{\partial (S \times \Omega)}{\partial S}$$

$$= \frac{(\dot{S} + S \times \Omega)}{|D \times S|^2} \left[I - \frac{2(D \times S)(D \times S)\cdot}{|D \times S|^2} \right] (D\times) - \frac{(D \times S)\cdot}{|D \times S|^2} (\Omega \times)$$

$$= \frac{(\dot{S} + S \times \Omega) \times D}{|D \times S|^2} - 2 \left[\left(\frac{\partial y_2}{\partial R} \right) \cdot (\dot{S} + S \times \Omega) \right] \frac{\partial y_2}{\partial R} \times D + \Omega \times \frac{\partial y_2}{\partial R} \tag{b}$$

The velocity gradient of azimuth rate is simply the position gradient of azimuth, found in A.1.2.

$$\frac{\partial \dot{y}_2}{\partial V} = \frac{\partial y_2}{\partial R} = \frac{D \times S}{|D \times S|^2} \tag{c}$$

Sensitivity of azimuth rate measurement to station location errors is found by interchanging the order of differentation (see A.1.2, equation (d)).

$$\frac{\partial \dot{y}_2}{\partial \Delta} = \frac{d}{dt}\left(\frac{\partial y_2}{\partial \Delta}\right)$$

$$= \frac{d}{dt}\left[-\frac{\partial y_2}{\partial R}\begin{pmatrix} N & E & D \end{pmatrix} + \left(\frac{-(S\cdot E)(S\cdot D)}{|R_T||D \times S|^2}, \frac{\tan(Lat)}{|R_T|} + \frac{(S\cdot N)(S\cdot D)}{|R_T||D \times S|^2}, 0\right)\right]$$

$$= \left[\frac{-\partial \dot{y}_2}{\partial R} + \Omega x \frac{\partial y_2}{\partial R}\right]\begin{pmatrix} N & E & D \end{pmatrix} + \frac{(\dot{S} + S \times \Omega)\cdot}{|R_T||D \times S|^2}\left\{D + 2\ (S\cdot D)Dx\frac{\partial y_2}{\partial R}\right\}(-S\cdot E, S\cdot N,\ 0)$$

$$+ \frac{S\cdot D}{|R_T||D \times S|^2}\left\{\dot{S} + S \times \Omega\right\}\cdot(-E,\ N,\ 0) \qquad \text{(d)}$$

After some considerable manipulation,

$$\frac{\partial \dot{y}_2}{\partial \Delta} = \left[-\frac{\partial \dot{y}_2}{\partial R} + \Omega x \frac{\partial y_2}{\partial R}\right]\cdot\begin{pmatrix} N & E & D \end{pmatrix} + \frac{1}{|RT||D + S|^2}\left\{[S\cdot D]\ (\dot{S} + S \times \Omega)\right.$$

$$\left. + \left[(\dot{S} + S \times \Omega)\cdot(D + 2\ (S\cdot D)\ Dx\frac{\partial y_2}{\partial R})\right]S\right\}\cdot(-E,\ N,\ 0)$$

Sensitivity of azimuth rate to time bias is taken to be the total time derivative of azimuth rate.

$$\frac{\partial \dot{y}_2}{\partial \tau} \equiv \frac{d\dot{y}}{dt} = \frac{d}{dt}\left\{\frac{\partial y_2}{\partial R}(\dot{S} + S \times \Omega)\right\} \qquad \text{(e)}$$

$$= \frac{\partial \dot{y}_2}{\partial R}(\dot{S} + S \times \Omega) + \frac{\partial y_2}{\partial R}(\ddot{S} + \dot{S} \times \Omega)$$

A.1.10 Elevation Rate, $\dot{y}_3$

This measurement is (from (e) of A.1.3)

$$\dot{y}_3 = \frac{\partial y_3}{\partial R}\left\{\dot{S} + S \times \Omega\right\} = \frac{(D \times S) \times S}{s^2\ |D \times S|}\cdot\left\{\dot{S} + S \times \Omega\right\} \qquad \text{(a)}$$

The position gradient of elevation rate, $\frac{\partial \dot{y}_3}{\partial R}$, is derived as follows:

$$\frac{\partial \dot{y}_3}{\partial R} = \frac{\partial \dot{y}_3}{\partial S} = \frac{\partial \dot{y}_3}{\partial (D \times S)} \quad \frac{\partial (D \times S)}{\partial S} + \frac{\partial \dot{y}_3}{\partial S} + \frac{\partial \dot{y}_3}{\partial (S \times \Omega)} \quad \frac{\partial (S \times \Omega)}{\partial S}$$

$$= \left\{\dot{S} + S \times \Omega\right\} \cdot \frac{(-Sx)}{s^2 |D \times S|} \left[I - \frac{(D \times S)(D \times S)}{|D \times S|^2}\right]$$

$$+ \left\{\dot{S} + S \times \Omega\right\} \cdot \frac{(D \times S)x}{s^2 |D \times S|} \left[I - \frac{2SS\cdot}{s^2}\right] + \frac{\Omega x \partial y_3}{\partial R}$$

$$= \left[\left\{\dot{S} + S \times \Omega\right\} x \ (D \times S) + \left(S \ x \left\{\dot{S} + S \times \Omega\right\}\right) xD\right] \frac{1}{s^2 |D \times S|^2} \qquad (b)$$

$$+ \left\{\dot{S} + S \times \Omega\right\} \cdot \frac{\partial y_3}{\partial R} \left[Dx \frac{\partial y_2}{\partial R} - 2 \frac{S}{s^2}\right] + \Omega x \frac{\partial y_3}{\partial R}$$

The velocity gradient of elevation rate is simply equal to the position gradient of elevation, derived in A.1.3

$$\frac{\partial \dot{y}_3}{\partial V} = \frac{\partial y_3}{\partial R} = \frac{(D \times S) \ xS}{s^2 |D \times S|} \qquad (c)$$

The sensitivity of the elevation rate measurement to station location errors is found by using the results of equation (d) of A.1.3 and interchanging the order of differentiation.

$$\frac{\partial \dot{y}_3}{\partial \Delta} = \frac{d}{dt} \left(\frac{\partial y_3}{\partial \Delta}\right)$$

$$= \frac{d}{dt} \left[- \frac{\partial y_3}{\partial R} \left(N \ E \ D\right) + \frac{S\cdot}{R_T \ |D \times S|} \left(N \ E \ O\right)\right] \qquad (d)$$

$$= \left[- \frac{\partial \dot{y}_3}{\partial R} + \Omega \times \frac{\partial y_3}{\partial R} \right] \begin{pmatrix} N & E & D \end{pmatrix}$$

$$+ \frac{1}{|R_T||D \times S|} \left\{ (\dot{S} + S \times \Omega) - \left[\frac{\partial y_2}{\partial R} \cdot Dx \; (\dot{S} + S \times \Omega) \right] S \right\} \cdot \begin{pmatrix} N & E & 0 \end{pmatrix}$$

The sensitivity of elevation rate to station time bias is given by

$$\frac{\partial y_3}{\partial \tau} \equiv \frac{d\dot{y}_3}{dt} = \frac{d}{dt} \left\{ \frac{\partial y_3}{\partial R} \cdot (\dot{S} + S \times \Omega) \right\}$$

$$= \frac{\partial \dot{y}_3}{\partial R} \cdot (\dot{S} + S \times \Omega) + \frac{\partial y_3}{\partial R} \cdot (\ddot{S} + \dot{S} \times \Omega) \qquad \text{(e)}$$

A.1.11 Right Ascension rate, $\dot{y}_4$

The measurement simulation is given in A.1.4 as

$$\dot{y}_4 = \frac{\partial y_4}{\partial R} \dot{S} = \frac{(K_e \times S)}{|K_e \times S|^2} \cdot \dot{S} \qquad \text{(a)}$$

The position gradient of right ascension rate is

$$\frac{\partial \dot{y}_4}{\partial R} = \frac{\partial \dot{y}_4}{\partial S} = \frac{\dot{S} \cdot}{|K_e \times S|^2} \left[I - \frac{2(K_e \times S)(K_e \times S) \cdot}{|K_e \times S|^2} \right] (K_e \times D)$$

$$= \frac{(\dot{S} \times K_e)}{|K_e \times S|^2} - 2 \left(\frac{\partial y_4}{\partial R} \cdot \dot{S} \right) \frac{\partial y_4}{\partial R} \times K_e \qquad \text{(b)}$$

The velocity gradient of right ascension rate is merely the position gradient of right ascension.

$$\frac{\partial \dot{y}_4}{\partial V} = \frac{\partial y_4}{\partial R} = \frac{K_e \times S}{|K_e \times S|^2} \qquad (c)$$

The sensitivity of right ascension rate to station location errors is found from the results of equation (d) of A.1.4 and interchange of the order of differentiation.

$$\frac{\partial \dot{y}_4}{\partial \Delta} = \frac{d}{dt}\left(\frac{\partial y_4}{\partial \Delta}\right) = \frac{d}{dt}\left[-\frac{\partial y_4}{\partial R}\left(N\ E\ D\right)\right]$$

$$= -\frac{d}{dt}\left(\frac{\partial y_4}{\partial R}\right)\left(N\ E\ D\right) - \frac{\partial y_4}{\partial R}\left(\dot{N}\ \dot{E}\ \dot{D}\right)$$

$$= -\left[\frac{\partial \dot{y}_4}{\partial R} - \Omega \times \frac{\partial y_4}{\partial R}\right]\left(N\ E\ D\right) \qquad (d)$$

The time bias sensitivity of right ascension rate is

$$\frac{\partial \dot{y}_4}{\partial \tau} \equiv \frac{d\dot{y}_4}{dt} = \frac{d}{dt}\left[\frac{\partial y_4}{\partial R} \cdot \dot{S}\right]$$

$$= \frac{\partial \dot{y}_4}{\partial R} \cdot \dot{S} + \frac{\partial y_4}{\partial R} \cdot \ddot{S} \qquad (e)$$

A.1.12 Declination rate, $\dot{y}_5$

The measurement of declination rate was derived in A.1.5 to be

$$\dot{y}_5 = \frac{\partial y_5}{\partial R} \cdot \dot{S} = \frac{S\times(K_e \times S)}{s^2|K_e \times S|} \cdot \dot{S} \qquad (a)$$

The position gradient of this measurement is

$$\frac{\partial \dot{y}_5}{\partial R} = \frac{\partial \dot{y}_5}{\partial S} = \frac{\partial \dot{y}_5}{\partial (K_e \times S)} \frac{\partial (K_e \times S)}{\partial S} + \frac{\partial \dot{y}_5}{\partial S}$$

$$= \frac{\dot{S}\cdot(Sx)}{s^2 |K_e \times S|}\left[I - \frac{(K_e \times S)(K_e \times S)}{|K_e \times S|^2}\right](K_e\ x)$$

$$+ \frac{\dot{S}\cdot(S \times K_e)x}{s^2 |K_e \times S|}\left[I - \frac{2SS\cdot}{s^2}\right]$$

$$= \frac{1}{s^2 |K_e \times S|}\left\{(\dot{S} \times S) \times K_e + \dot{S}x\ (S \times K_e)\right\}$$

$$- \dot{S}\cdot \frac{\partial y_5}{\partial R}\left\{\frac{\partial y_4}{\partial R} \times K_e + 2\frac{S}{s^2}\right\} \qquad \text{(b)}$$

The velocity gradient of declination rate is the position gradient of declination, found in A.1.5.

$$\frac{\partial \dot{y}_5}{\partial V} = \frac{\partial y_5}{\partial R} = \frac{S_x(K_e \times S)}{s^2 |K_e \times S|} \qquad \text{(c)}$$

The sensitivity of this measurement to station location errors is easily found to be

$$\frac{\partial \dot{y}_5}{\partial \Delta} = \frac{d}{dt}\left(\frac{\partial y_5}{\partial \Delta}\right) = \frac{d}{dt}\left[\frac{-\partial y_5}{\partial R}\begin{pmatrix} N & E & D \end{pmatrix}\right] = \left[\frac{-\partial \dot{y}_5}{\partial R} + \Omega x \frac{\partial y_5}{\partial R}\right]\begin{pmatrix} N & E & D \end{pmatrix} \qquad \text{(d)}$$

The time bias partial derivative of declination rate is

$$\frac{\partial \dot{y}_5}{\partial \tau} \equiv \frac{d\dot{y}_5}{dt} = \frac{d}{dt}\left[\frac{\partial y_5}{\partial R}\dot{S}\right] = \frac{\partial \dot{y}_5}{\partial R}\cdot \dot{S} + \frac{\partial y_5}{\partial R}\cdot \ddot{S} \qquad \text{(e)}$$

A.1.13 1-Direction Cosine rate, $\dot{y}_6$

The formulation, for this measurement was derived in A.1.6, equation (e).

$$\dot{y}_6 = \frac{\partial y_6}{\partial R} (\dot{S} + S \times \Omega) = \frac{S \times (N \times S)}{s^3} \cdot (\dot{S} + S \times \Omega) \qquad \text{(a)}$$

The position gradient is found to be

$$\frac{\partial \dot{y}_6}{\partial R} = \frac{\partial \dot{y}_6}{\partial S} = \frac{\partial \dot{y}_6}{\partial (N \times S)} \frac{\partial (N \times S)}{\partial S} + \frac{\partial \dot{y}_6}{\partial S} + \frac{\partial \dot{y}_6}{\partial (S \times \Omega)} \frac{\partial (S \times \Omega)}{\partial S}$$

$$= \frac{N \times [S \times (\dot{S} + S \times \Omega)]}{s^3} + (\dot{S} + S \times \Omega) \cdot \frac{(S \times N) \times}{s^3} \left[I - \frac{3SS}{s^2} \right] + \frac{\partial y_6}{\partial R} (-\Omega \times)$$

$$= \frac{N \times [S \times (\dot{S} + S \times \Omega)]}{s^3} + \frac{(N \times S) \times (\dot{S} + S \times \Omega)}{s^3}$$

$$- \frac{3S}{s^2} \left[\frac{\partial y_6}{\partial R} \cdot (\dot{S} + S \times \Omega) \right] + \Omega \times \frac{\partial y_6}{\partial R} \qquad \text{(b)}$$

The velocity gradient of this measurement is

$$\frac{\partial \dot{y}_6}{\partial \dot{\Delta}} = \frac{\partial y_6}{\partial R} = \frac{S \times (N \times S)}{s^3} \qquad \text{(c)}$$

The sensitivity of 1-direction cosine rate to station location errors is

$$\frac{\partial \dot{y}_6}{\partial \Delta} = \frac{d}{dt} \left(\frac{\partial y_6}{\partial \Delta} \right)$$

$$= \frac{d}{dt}\left[\frac{\partial y_6}{\partial R}\begin{pmatrix} E & N & U \end{pmatrix} - \frac{1}{s|R_T|}\ (-S \cdot D,\ S \cdot E \text{ Tan Lat},\ 0)\right]$$

$$= \left[-\frac{\partial \dot{y}_6}{\partial R} + \Omega x \frac{\partial y_6}{\partial R}\right]\begin{pmatrix} N & E & D \end{pmatrix}$$

$$- \left[(\dot{S} + S \times \Omega) - \frac{(S \cdot \dot{S})}{s^2}\, S\right] \cdot \left(\frac{-D,\ E \tan \text{Lat},\ 0}{s|R_T|}\right) \tag{d}$$

The time bias sensitivity is

$$\frac{\partial \dot{y}_6}{\partial \tau} \equiv \frac{d\dot{y}_6}{dt} = \frac{\partial \dot{y}_6}{\partial R} \cdot (\dot{S} + S \times \Omega) + \frac{\partial y_6}{\partial R} \cdot (\ddot{S} + \dot{S} \times \Omega) \tag{e}$$

A.1.14 m-Direction Cosine rate, $\dot{y}_7$

Derivations for this quantity follow those for the 1-direction cosine rate in A.1.13.

$$\dot{y}_7 = \frac{\partial y_7}{\partial R}\ (\dot{S} + S \times \Omega) = \frac{S \times (E \times S)}{s^3} \cdot (\dot{S} + S \times \Omega) \tag{a}$$

$$\frac{\partial \dot{y}_7}{\partial R} = \frac{E \times [S \times (\dot{S} + S \times \Omega)]}{s^3} + \frac{(E \times S) \times (\dot{S} + S \times \Omega)}{s^3}$$

$$- \frac{3S}{s^2}\left[\frac{\partial y_7}{\partial R} \cdot (\dot{S} + S \times \Omega)\right] + \Omega \times \frac{\partial y_7}{\partial R} \tag{b}$$

$$\frac{\partial \dot{y}_7}{\partial V} = \frac{\partial y_7}{\partial R} = \frac{S \times (E \times S)}{s^3} \tag{c}$$

Sensitivity to station location errors is

$$\frac{\partial \dot{y}_7}{\partial \Delta} = \frac{d}{dt}\left[- \frac{\partial y_7}{\partial \Delta}\left(N\ E\ D\right) + \frac{S\cdot}{s|R_T|}\ (0,\ N \tan \mathrm{Lat}\ D,\ 0)\right]$$

$$= \left[- \frac{\partial \dot{y}_7}{\partial R} + \Omega \times \frac{\partial y_7}{\partial R}\right]\left(N\ E\ D\right)$$

$$+ \left[(\dot{S} + S \times \Omega) - \frac{(S\cdot \dot{S})S}{s^2}\right] \cdot \frac{(0,\ N \tan \mathrm{Lat},\ D \qquad 0)}{s|R_T|} \qquad \text{(d)}$$

The sensitivity to a time bias is

$$\frac{\partial \dot{y}_7}{\partial \tau} = \frac{d}{dt}\ \dot{y}_7 = \frac{\partial \dot{y}_7}{\partial R}\ (\dot{S} + S \times \Omega) + \frac{\partial y_6}{\partial R}\ (\ddot{S} + S \times \Omega) \qquad \text{(e)}$$